ISSN 2348-1196 (print)
International Journal of Computer Science and Information Technology Research ISSN 2348-120X (online)
Vol. 10, Issue 1, pp: (38-42), Month: January - March 2022, Available at: www.researchpublish.com

Some Applications of Integration by Parts for Fractional Calculus

Chii-Huei Yu

School of Mathematics and Statistics,

Zhaoqing University, Guangdong, China

Abstract: **In this paper, we make use of the integration by parts for fractional calculus to solve some fractional integrals based on Jumarie type of modified Riemann-Liouville (R-L) fractional derivative. A new multiplication of fractional analytic functions plays an important role in this study. In fact, the method we used is a natural generalization of the integration by parts for classical calculus.**

Keywords: **integration by parts for fractional calculus, fractional integrals, Jumarie type of modified R-L fractional derivative, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus is a mathematical analysis tool used to study arbitrary order derivatives and integrals. It unifies and extends the concepts of integer order derivatives and integrals [1-5]. Generally, many scientists do not know these fractional integrals and derivatives, and they have not been used in pure mathematical context until recent years. However, in the past few decades, the fractional integrals and derivatives have frequently appeared in many scientific fields such as fluid mechanics, viscoelasticity, physics, image processing, economics and engineering [6-13].

Until now, the definition of fractional derivative is not unique. The commonly used definitions are Riemann-Liouvellie (R-L) fractional derivative, Caputo definition of fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [1-5]. Jumarie's modification of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function [14-15].

In this article, we use integration by parts for fractional calculus to evaluate several fractional integrals based on Jumarie's modification of R-L fractional derivative. Some fractional analytic functions are introduced such as the fractional exponential function, cosine function, sine function, and logarithmic function. A new multiplication of fractional analytic functions plays an important role in this research. In fact, the method we used is the natural generalization of integration by parts for classical calculus.

II. DEFINITIONS AND PROPERTIES

At First, the fractional calculus used in this paper is introduced below.

Definition 2.1: Suppose that α is a real number, and p is a positive integer. The Jumarie type of modified Riemann-Liouville fractional derivative [15] is defined as

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}(x-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}(x-\tau)^{-\alpha}[f(\tau)-f(x_0)]\,d\tau & \text{if } 0\le\alpha<1 \\ \frac{d^p}{dx^p}\left({}_{x_0}D_x^{\alpha-p}\right)[f(x)], & \text{if } p\le\alpha<p+1 \end{cases} \quad (1)$$

where $\Gamma(\ )$ is the gamma function. Moreover, we define the α-fractional integral of $f(x)$ by $\left({}_{x_0}I_x^\alpha\right)[f(x)] = \left({}_{x_0}D_x^{-\alpha}\right)[f(x)]$, where $\alpha > 0$. If $\left({}_{x_0}I_x^\alpha\right)[f(x)]$ exists, then $f(x)$ is called an α-fractional integrable function. We have the following properties.

Proposition 2.2: *If α, β, c are real numbers and $\beta \geq \alpha > 0$, then*

$$ {}_0D_x^\alpha\left[x^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{2}$$

and

$$ {}_0D_x^\alpha[c] = 0. \tag{3}$$

Next, the fractional analytic function is defined.

Definition 2.3 ([16])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, $0 < \alpha \leq 1$. Suppose that the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha}$ on some open interval $(x_0 - r, x_0 + r)$, then $f_\alpha(x^\alpha)$ is called α-fractional analytic at x_0, where r is the radius of convergence about x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

A new multiplication of fractional analytic functions is introduced below.

Definition 2.4 ([18])**:** Assume that $0 < \alpha \leq 1$, $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions,

$$ f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha}, \tag{4}$$

$$ g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} .$$

(5)We define

$$ f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) $$
$$ = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} $$
$$ = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) x^{k\alpha}. \tag{6}$$

In the following, we provide some fractional analytic functions.

Definition 2.5 ([17, 18])**:** The Mittag-Leffler function is defined by

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{7}$$

where α is a real number, $\alpha \geq 0$, and z is a complex number.

Definition 2.6 ([19])**:** Let $0 < \alpha \leq 1$, and p, x be real numbers. $E_\alpha(px^\alpha) = \sum_{k=0}^{\infty} \frac{p^k x^{k\alpha}}{\Gamma(k\alpha+1)}$ is called α-fractional exponential function, and the α-fractional cosine, sine and logarithmic function are defined as follows:

$$ cos_\alpha(px^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k} x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{8}$$

$$ sin_\alpha(px^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k p^{2k+1} x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{9}$$

and

$$ Ln_\alpha(x^\alpha) = \left({}_1I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -1}\right]. \tag{10}$$

Proposition 2.7 (fractional Euler's formula)**:** *Let $0 < \alpha \leq 1$, then*

$$ E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha) . \tag{11}$$

Theorem 2.8 (integration by parts for fractional calculus) ([20])**:** *Assume that* $0<\alpha\leq 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions, *then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha)\otimes\left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]=\left[f_\alpha(x^\alpha)\otimes g_\alpha(x^\alpha)\right]_{x=a}^{x=b}-\left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha)\otimes\left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \qquad (12)$$

III. EXAMPLES

In the following, we use integration by parts for fractional calculus to evaluate some fractional integrals.

Example 3.1: Let $0<\alpha\leq 1$, then by integration by parts for fractional calculus, we have

$$\left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}\otimes E_\alpha(2x^\alpha)\right]$$

$$=\left[\frac{1}{\Gamma(3\alpha+1)}x^{3\alpha}\otimes\frac{1}{2}E_\alpha(2x^\alpha)\right]_0^x-\left({}_0I_x^\alpha\right)\left[\frac{1}{2}E_\alpha(2x^\alpha)\otimes\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\right]$$

$$=\frac{1}{2\Gamma(3\alpha+1)}x^{3\alpha}\otimes E_\alpha(2x^\alpha)-\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}\otimes E_\alpha(2x^\alpha)\right]$$

$$=\frac{1}{2\Gamma(3\alpha+1)}x^{3\alpha}\otimes E_\alpha(2x^\alpha)-\frac{1}{4\Gamma(2\alpha+1)}x^{2\alpha}\otimes E_\alpha(2x^\alpha)+\frac{1}{4}\left({}_0I_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\otimes E_\alpha(2x^\alpha)\right]$$

$$=\frac{1}{2\Gamma(3\alpha+1)}x^{3\alpha}\otimes E_\alpha(2x^\alpha)-\frac{1}{4\Gamma(2\alpha+1)}x^{2\alpha}\otimes E_\alpha(2x^\alpha)+\frac{1}{8\Gamma(\alpha+1)}x^{\alpha}\otimes E_\alpha(2x^\alpha)-\frac{1}{8}\left({}_0I_x^\alpha\right)[E_\alpha(2x^\alpha)]$$

$$=\left[\frac{1}{2\Gamma(3\alpha+1)}x^{3\alpha}-\frac{1}{4\Gamma(2\alpha+1)}x^{2\alpha}+\frac{1}{8\Gamma(\alpha+1)}x^{\alpha}-\frac{1}{16}\right]\otimes E_\alpha(2x^\alpha)+\frac{1}{16}. \qquad (13)$$

Example 3.2: Let $0<\alpha\leq 1$. Since

$$\left({}_0I_x^\alpha\right)[E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)]$$

$$=\left[\frac{1}{4}E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)\right]_0^x+\left({}_0I_x^\alpha\right)\left[\frac{1}{4}E_\alpha(4x^\alpha)\otimes 3sin_\alpha(3x^\alpha)\right]$$

$$=\frac{1}{4}E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)-\frac{1}{4}+\frac{3}{4}\left[\left({}_0I_x^\alpha\right)\left[\frac{1}{4}E_\alpha(4x^\alpha)\otimes 3sin_\alpha(3x^\alpha)\right]\right]$$

$$=\frac{1}{4}E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)-\frac{1}{4}+\frac{3}{4}\left[\frac{1}{16}E_\alpha(4x^\alpha)\otimes 3sin_\alpha(3x^\alpha)-\left({}_0I_x^\alpha\right)\left[\frac{1}{16}E_\alpha(4x^\alpha)\otimes 9cos_\alpha(3x^\alpha)\right]\right]$$

$$=\frac{1}{4}E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)-\frac{1}{4}+\frac{9}{64}E_\alpha(4x^\alpha)\otimes sin_\alpha(3x^\alpha)-\frac{27}{64}\left({}_0I_x^\alpha\right)[E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)]. \qquad (14)$$

It follows that

$$\left({}_0I_x^\alpha\right)[E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)]$$

$$=\frac{64}{91}\left[\frac{1}{4}E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)-\frac{1}{4}+\frac{9}{64}E_\alpha(4x^\alpha)\otimes sin_\alpha(3x^\alpha)\right]$$

$$=\frac{16}{91}E_\alpha(4x^\alpha)\otimes cos_\alpha(3x^\alpha)+\frac{9}{91}E_\alpha(4x^\alpha)\otimes sin_\alpha(3x^\alpha)-\frac{16}{91}. \qquad (15)$$

Example 3.3: Let $0<\alpha\leq 1$, and n be a non-negative integer. Then

$$\left({}_1I_x^\alpha\right)\left[\frac{1}{\Gamma(n\alpha+1)}x^{n\alpha}\otimes Ln_\alpha(x^\alpha)\right]$$

$$=\left[\frac{1}{\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha}\otimes Ln_\alpha(x^\alpha)\right]_1^x-\left({}_1I_x^\alpha\right)\left[\frac{1}{\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha}\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right]$$

$$=\frac{1}{\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha}\otimes Ln_\alpha(x^\alpha)-\left({}_1I_x^\alpha\right)\left[\frac{1}{(n+1)\cdot\Gamma(n\alpha+1)}x^{n\alpha}\right]$$

$$=\frac{1}{\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha}\otimes Ln_\alpha(x^\alpha)-\frac{1}{(n+1)\cdot\Gamma((n+1)\alpha+1)}x^{(n+1)\alpha}+\frac{1}{(n+1)\cdot\Gamma((n+1)\alpha+1)}. \qquad (16)$$

Example 3.4: If $0<\alpha\leq 1$. Since

$$\left({}_{1}I_{x}^{\alpha}\right)\left[Ln_{\alpha}(x^{\alpha})\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right]$$
$$=[Ln_{\alpha}(x^{\alpha})\otimes Ln_{\alpha}(x^{\alpha})]_{1}^{x}-\left({}_{1}I_{x}^{\alpha}\right)\left[Ln_{\alpha}(x^{\alpha})\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right]$$
$$=\left(Ln_{\alpha}(x^{\alpha})\right)^{\otimes 2}-\left({}_{1}I_{x}^{\alpha}\right)\left[Ln_{\alpha}(x^{\alpha})\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right].$$

It follows that

$$\left({}_{1}I_{x}^{\alpha}\right)\left[Ln_{\alpha}(x^{\alpha})\otimes\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes -1}\right]=\frac{1}{2}\left(Ln_{\alpha}(x^{\alpha})\right)^{\otimes 2}. \quad (17)$$

IV. CONCLUSION

The purpose of this paper is to solve several fractional integrals by using integration by parts for fractional calculus. The method we used is a natural generalization of integration by parts for classical calculus. The new multiplication we defined plays an important role in this article, and it is the natural operation in fractional calculus. In the future, we will use Jumarie's modified R-L fractional derivative to study the problems in fractional calculus and fractional differential equations.

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